T. Y. B. Sc. (Mathematics)

20 Problems on Uniform Convergence with Hints

MT-342: Real Analysis II

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1. Let f_n be a sequence of continuous real-valued functions that converges uniformly on $[a, b]$. Let

$$F_n(x) = \int_a^x f_n(t)dt, \quad a \leq x \leq b$$

Show that F_n converges uniformly on $[a, b]$. (Hint: Use Cauchy criterion for uniform convergence and then use modulus of integral $\leq$ integral of modulus)

2. Let f_n be a sequence of continuous functions on $[0, 1]$ that converges uniformly on $[0, 1]$.
(a) Show that there exists $M > 0$ such that $|f_n(x)| \leq M, \ \forall n$ and $\forall x \in [0, 1]$.
(Hint: Use Cauchy criterion, then triangle inequality and then choose maximum of a finite set.)
(b) Does the result in (a) hold if uniform convergence is replaced by pointwise convergence ? (Hint: Try $\frac{1}{1+nx}$).

3. Show by an example that Dini's theorem is no longer true if we omit the hypothesis of M that M is compact.
Let us recall Dini's theorem for a ready reference.
Dini's theorem: Let f_n be a sequence of continuous real-valued functions on a compact metric space (M, ρ) such that

$$f_1 \leq f_2 \leq \ldots f_n \leq \ldots, \text{on } M.$$

If $f_n \to f$ pointwise to a continuous function f, then $f_n \to f$ uniformly on M.
(Hint: Consider $f_n(x) = \frac{x}{n}$ on $[0, \infty)$).

4. Let A be a dense subset of a metric space M and let $f_n \to f$ uniformly on A. Prove that $f_n \to f$ uniformly on M.
(Hint : Combine definitions of uniform convergence and denseness).

5. Let f_n be a sequence of functions converging uniformly to a continuous function f on $[0, \infty)$. Prove that

$$\lim_{n\to\infty} f_n(x + \frac{1}{n}) = f(x), \quad 0 \leq x < \infty$$

(Hint: Use definition of uniform convergence to get N_1, then use definition of continuity to get a δ and hence N_2. Take $\max\{N_1, N_2\}$).

6. Give an example in each of the following cases :
(a) $f_n \to f$ on $[0, 1]$, each f_n is Riemann integrable on $[0, 1]$, but f is not Riemann integrable on $[0, 1]$. (Try: $f_n = \chi_{Q_n}$, where $Q_n = \{r_1, r_2, \ldots, r_n\}$ and $r_1, r_2, \ldots, r_n \ldots$ is enumeration of rational numbers in $[0, 1]$.
(b) $f_n \to f$ uniformly on $\mathbb{R}$ each f_n is differentiable on $\mathbb{R}$, but f is not differentiable. (Try: $f_n = \sqrt{x^2 + \frac{1}{n}}$).
(c) $f_n \to f$ uniformly on $\mathbb{R}$ each f_n is differentiable on $\mathbb{R}$, but f_n' is not (even pointwise) convergent. (Hint: Take $f_n(x) = \frac{\sin nx}{n}$).
(d) $f_n \to f$ uniformly on $[0, 1]$ each f_n is differentiable on $[0, 1]$, f is differentiable on $[0, 1]$, f_n' is convergent but f_n' does not converge to f'. (Hint : Try $\frac{x^n}{n}$).

7. Let f_n be a sequence of continuous functions converging uniformly to f on $[a, b]$. Let g be a continuous function on $[a, b]$. Prove that

$$\lim_{n\to\infty} \int_a^b f_n g = \int_a^b fg$$

(Hint: Estimate $\left| \int_a^b (f_n g - fg) \right| \leq \int_a^b |(f_n g - fg)|$ and use definition of uniform convergence).

8. If the series $\sum_{n=0}^{\infty} a_n$ is convergent and

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad -1 < x < 1$$

then prove that f is continuous on $(0,1)$. (Hint: The power series $\sum_{n=0}^{\infty} a_n x^n$ is given to be convergent at $x = 1$. Thus it is uniformly convergent on compact subsets of $(-1,1)$. Now use that each term $a_n x^n$ is continuous).

9. Let $u_1, u_2, \ldots$ be continuous functions on a metric space M. If $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent on a dense subset of M, prove that $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent on M.
(Use Problem 4 for the sequence of partial sums).

10. If $\sum_{n=0}^{\infty} |a_n|$ is convergent, then prove that

$$\int_0^1 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{a_n}{n+1}$$

is convergent (Hint: The power series $\sum_{n=0}^{\infty} a_n x^n$ is given to be convergent at $x = 1$. Thus it is uniformly convergent on compact subsets of $(-1,1)$. Thus term by term integration is valid on $(0,1)$.)

11. Without finding the sum $f(x)$ of the series

$$1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \quad -\infty < x < \infty,$$

prove that $f'(x) = 2xf(x)$ (Hint: Show that the radius of convergence of the series in ∞. Thus the series converges uniformly on any compact subset on $\mathbb{R}$. therefore the term by term differentiation is valid).

12. Justify whether true or false : If f_n converges uniformly and g_n converges uniformly, then $f_n g_n$ converges uniformly
(Check $f_n = g_n = x + \frac{1}{n}$ on $\mathbb{R}$) .

13. Let $f_n(x) = \left(1 + \dfrac{x}{n}\right)^n$, $x \in \mathbb{R}$. Show that f_n converges uniformly to e^x on any compact subset $[a, b] \subset \mathbb{R}$ (Hint : Apply Dini's theorem).

14. Show that the sequence $\dfrac{x^n}{1 + x^2 n}$ converges uniformly on $[2, 10]$.
(Put upper bound on f_n by using bounds of $[2, 10]$.)
Does the sequence converge on $[0, 2]$? What about $[-2, 0]$?
(Check what happens at $x = 1$ or $x = -1$.)

15. Show that the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$ is converges uniformly, but not absolutely on $[0, 1)$. (Hint: Show that the sequence of partial sums is uniformly Cauchy but the same is not true with the sequence of partial sums after taking absolute values).

16. Give an example of a series $\sum f_n(x)$ such that each f_n is continuous on $\mathbb{R}$, but the sum is not continuous on $\mathbb{R}$. (Try $f_n(x) = \dfrac{x^2}{(1 + x^2)^n}$) (Watch the sum at $x = 0$).

17. Show that the sequence $\dfrac{nx}{1 + n^2 x^2}$ is not uniformly convergent on any interval containing 0.
(Hint: what happens at $x = \frac{1}{n}$).

18. Test the uniform convergence of the series $\displaystyle\sum_{n=1}^{\infty} x e^{-nx}, 0 \leq x \leq 1$.
(Find $\sup\{|f_n(x) - f(x)|\}$).

19. Test the uniform convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{x}{n(1 + nx^2)}$ on $\mathbb{R}$.
(Hint: Differentiate, find upper bound on f_n and use Weierstrass M-test).

20. Test for uniform convergence, the series

$$\frac{2x}{1 + x^2} + \frac{4x^3}{1 + x^4} + \frac{8x^7}{1 + x^8} + \cdots, \quad \frac{-1}{2} < x < \frac{1}{2}$$

(Use Weierstrass M-test).